Charge Fluctuations in Finite Coulomb Systems

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Abstract

When described in a grand canonical ensemble, a finite Coulomb system exhibits charge fluctuations. These fluctuations are studied in the case of a classical (i.e. non-quantum) system with no macroscopic average charge. Assuming the validity of macroscopic electrostatics gives, on a three-dimensional finite large conductor of volume V, a mean square charge $< Q^2 >$ which goes as $V^{1/3}$. More generally, in a short-circuited capacitor of capacitance C, made of two conductors, the mean square charge on one conductor is $< Q^2 >= TC$, where T is the temperature and C the capacitance of the capacitor. The case of only one conductor in a grand canonical ensemble is obtained by removing the other conductor to infinity. The general formula is checked in the weak-coupling (Debye-Hückel) limit for a spherical capacitor. For two-dimensional Coulomb systems (with logarithmic interactions), there are exactly solvable models which reveal that, in some cases, macroscopic electrostatics is not applicable even for large conductors. This is when the charge fluctuations involve only a small number of particles. The mean square charge on one two-dimensional system alone, in the grand canonical ensemble, is expected to be, at most, one squared elementary charge.

KEY WORDS: Finite Coulomb systems; charge fluctuations.

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1 INTRODUCTION

This paper is dedicated to Michael Fisher on the occasion of his 70th birthday. Part of it deals with the Debye-Hückel theory of Coulomb systems, to which Michael brought elaborate refinements, in particular insisting on the importance of taking hard cores into account. I apologize for using here only the simple point-particle version.

I consider the classical (i.e. non-quantum) equilibrium statistical mechanics of Coulomb systems: systems of charged particles interacting through the Coulomb law (plus perhaps short-ranged forces), such as plasmas or electrolytes. Such a system, when described by a grand canonical ensemble, is expected to exhibit charge fluctuations. The aim of the present paper is to study these fluctuations, for a finite but macroscopic system.

The grand canonical ensemble is often introduced by considering a system in contact with a surrounding infinite system (the reservoir), with possible exchanges of energy and particles between the system and the reservoir. In this approach, it is assumed that the energy of interaction between the system and the reservoir is negligible. This is indeed the case for a finite but macroscopic system, when the interparticle forces are short-ranged (then, the system energy goes as its volume, while the interaction energy goes only as its surface area). However, this approach has to be modified when there are long-ranged forces such as Coulomb ones. Then, for the interaction energy between the system and the reservoir be disregarded, is is necessary to assume that the reservoir is infinitely far away from the system under consideration. This is how the grand canonical ensemble will be defined in the following.

Studying the charge fluctuations in a given large subvolume Λ of an infinite Coulomb system (i.e. assuming that the reservoir is in contact with the subvolume) is a different problem, which has already been studied and solved[1, 2]. It was found that the mean square charge $\langle Q^2 \rangle$ in Λ behaves as its surface area S (not its volume V). In the presently studied geometry (infinitely remote reservoir), it will be argued that $\langle Q^2 \rangle$ is even weaker, behaving as $V^{1/3}$, for a 3-dimensional system. This behavior had been conjectured by Lieb and Lebowitz[3], but they could not prove it rigorously, i.e. by a purely microscopic argument. Here, on the contrary, our starting point will be the validity of macroscopic electrostatics of conductors, assumed without proof.

In general, in the grand canonical ensemble, the state of a system, made of s species of particles, depends on s chemical potentials. However, in the case of a Coulomb system, in the thermodynamic limit, the state depends only on $s - 1$ chemical potentials and is the same as in a grand canonical ensemble restricted to neutral configurations[3]. In the case of a macroscopic but finite Coulomb system, there may be a non-zero average charge $\langle Q \rangle$ and one more chemical potential is needed for controlling it. Here, we only consider the simple case in which there is only one reservoir, which is an infinite Coulomb system of the same nature as the system under consideration, and $\langle Q \rangle$ is expected to vanish if macroscopic electrostatics holds.

We shall first assume that our system and this reservoir are at some distance of each other. Since they can freely exchange particles, the system and the reservoir can be considered as two conductors forming a short-circuited capacitor. In Section 2, a general simple expression for the mean square charge of the system in that configuration will be derived; the case of the system in the grand canonical ensemble will be obtained by

removing the reservoir to infinity. In Section 3, these general results will be checked by a microscopic calculation in the weak-coupling (Debye-Hückel) limit.

Coulomb systems can be mimicked in a two-dimensional world, in which the Coulomb potential $1/r$ must be replaced by the two-dimensional solution of the Poisson equation $-\ln r$. Working in two dimensions has the advantage that exactly solvable models for the statistical mechanics of Coulomb systems are available. However, in two dimensions, some specific subtle points arise and deserve a separate discussion. In Section 4, two-dimensional models will be considered, and cases when macroscopic electrostatics cannot be used will be exhibited.

Rather than starting with a capacitor made of two conductors and removing one of them to infinity, one might want to study directly the case of one finite system in the grand canonical ensemble. How to formulate the Debye-Hückel theory in a finite system is discussed in Appendix B.

2 CHARGE FLUCTUATIONS IN A CAPACITOR

Let us consider a capacitor, made of two macroscopic conductors A and B separated by vacuum, with B surrounding A. Let C be the capacitance. Let Q be the charge on conductor A, $-Q$ the charge on conductor B. When the capacitor is short-circuited, the average charge $\langle \pm Q \rangle$ on each conductor vanishes. If the conductors are describable by classical (non-quantum) statistical mechanics, we claim that the mean square charge on each conductor is

$$\langle Q^2 \rangle = TC \tag{2.1}$$

where T is the temperature (in units of energy). This relation is a special case of Nyquist's formula[4] which gives the electrical current fluctuations in a linear electric circuit. The derivation of Nyquist's formula relies on the fluctuation-dissipation theorem in the theory of linear response. In the present case, a more direct derivation of eq.(2.1) can be made by using the simpler classical static linear response theory, as follows.

Let us introduce some given external charges which create an infinitesimal electric potential difference between the two conductors: let the potential on conductor A minus the potential on conductor B be $\delta\phi$ (for instance, in the case of a spherical capacitor, we introduce between the conductors two spherical concentric layers carrying opposite uniformly distributed charges; then, these layers create a potential which is zero on the outer conductor B and has some constant value $\delta\phi$ on the inner conductor A). The corresponding change in the Hamiltonian is $H' = \delta\phi\, Q$, where Q is the charge on conductor A. The short-circuited capacitor will respond by tranfering a charge $\delta Q = -C\,\delta\phi$ from conductor B to conductor A, in such a way that the total potential difference (external plus induced) between the conductors vanishes. Now, linear response theory says

$$-C\,\delta\phi = \delta Q = -\beta(\langle H'Q \rangle - \langle H' \rangle \langle Q \rangle) = -\beta\,\delta\phi \langle Q^2 \rangle \tag{2.2}$$

where β is the inverse temperature $1/T$ and the averages $\langle \ldots \rangle$ are taken in the absence of the perturbation H' (we have used $\langle Q \rangle = 0$). Eq.(2.2) proves eq.(2.1).It should be noted that C will be the usual capacitance for a given geometry only provided that macroscopic

electrostatics is applicable. A necessary condition is that the sizes of the conductors and the separation between them be large compared to the microscopic scale.

The mean square charge of a finite macroscopic Coulomb system in the grand canonical ensemble is obtained by sending conductor B to infinity. Then the mean square charge on the macroscopic body A is given by (2.1), where now C is the capacitance of the macroscopic body A alone. For a 3-dimensional system, this capacitance goes as $V^{1/3}$ where V is the volume, as stated in the Introduction.

In the special simple case of a spherical capacitor, made of an inner conductor of radius R_1 and an outer conductor of radius R_2, the capacitance is

$$C = \frac{1}{(1/R_1) - (1/R_2)} \tag{2.3}$$

It becomes R_1 if $R_2 \to \infty$, giving the mean square charge in a macroscopic but finite spherical Coulomb system of radius R, in the grand-canonical ensemble,

$$\langle Q^2 \rangle = TR \tag{2.4}$$

in agreement with previous findings about the charge correlations. [6]

Another limit of interest is when R_1 becomes large compared to $W = R_2 - R_1$ which keeps a fixed (macroscopic) value. One then obtains a plane capacitor with plate areas $S = 4\pi R_1^2$ and plate separation W, and indeed (2.3) becomes $C = S/(4\pi W)$, giving for the mean square charge per unit area on one plate

$$\lim_{S\to\infty} \frac{\langle Q^2 \rangle}{S} = \frac{T}{4\pi W} \tag{2.5}$$

3 WEAK-COUPLING LIMIT

We consider the simple geometry of a spherical capacitor made with a classical Coulomb fluid: There are two concentric spheres, centered at the origin, of radii R_1 and R_2 ($R_2 > R_1$). The shell between the spheres is empty. The ball of radius R_1 is filled with the fluid, as well as the whole space outside the sphere of radius R_2. The short-circuiting of the capacitor is described by assuming that the two filled regions are allowed to freely exchange particles.

The weak-coupling limit is a high-temperature one which is expected to be correctly described by the Debye-Hückel theory. The Coulomb fluid is made of several species of particles of number densities n_a and charges q_a. For the system to be stable, in addition to the Coulomb forces, there should be some short-ranged repulsive forces, but the weak-coupling limit can also be viewed as a low-density one in which these short-ranged forces can be neglected. Strictly speaking, the number densities are position-dependent near the fluid boundaries. However, taking this into account would only give corrections of higher order and therefore we consider the densities as constants. The Debye wave number is defined as $\kappa = (4\pi\beta \sum_a n_a q_a^2)^{1/2}$. Let $\rho(\mathbf{r})$ be the microscopic charge density at $\mathbf{r}$. We shall need the charge correlation function

$$\langle \rho(\mathbf{r})\rho(\mathbf{r}') \rangle = \sum_{ab} n_a q_a^2 n_b q_b^2 K(\mathbf{r}, \mathbf{r}') + \sum_a n_a q_a^2 \delta(\mathbf{r} - \mathbf{r}') \tag{3.1}$$

where both $\mathbf{r}$ and $\mathbf{r}'$ are in a filled region. $K(\mathbf{r},\mathbf{r}')$ is the solution of the partial differential equation

$$[\Delta - \kappa^2(\mathbf{r})]K(\mathbf{r},\mathbf{r}') = 4\pi\beta\delta(\mathbf{r}-\mathbf{r}') \tag{3.2}$$

where the source point $\mathbf{r}'$ is assumed to be in a filled region, while $\mathbf{r}$ can be anywhere: $\kappa^2(\mathbf{r}) = \kappa^2$ if $\mathbf{r}$ is in a filled region and $\kappa^2(\mathbf{r}) = 0$ if $\mathbf{r}$ is in the empty region. K and its normal derivative must be continuous at the boundaries $r = R_1$ and $r = R_2$, and $K \to 0$ as $r \to \infty$.

In the present Debye-Hückel scheme, the average charge on the sphere of radius R_1 vanishes, while the mean square charge on that sphere is

$$\langle Q^2 \rangle = \int_{r<R_1} \mathrm{d}\mathbf{r} \int_{r'<R_1} \mathrm{d}\mathbf{r}' \langle \rho(\mathbf{r})\rho(\mathbf{r}')\rangle \tag{3.3}$$

The solution K of (3.2) is studied in Appendix A. When used in (3.1) and (3.3) it gives

$$\beta\langle Q^2 \rangle = \frac{[1+\kappa R_2][\kappa R_1 \cosh(\kappa R_1) - \sinh(\kappa R_1)]}{\kappa[(1+\kappa R_2 - \kappa R_1)\cosh(\kappa R_1) + \sinh(\kappa R_1)]} \tag{3.4}$$

If both R_1 and R_2 are macroscopic, i.e. if $\kappa R_1, \kappa R_2 \gg 1$, (3.4) becomes

$$\beta\langle Q^2 \rangle = \frac{\kappa R_1 R_2}{2+\kappa R_2 - \kappa R_1} \tag{3.5}$$

The mean square charge in a large spherical subdomain of an infinite Coulomb system is retrieved by taking $R_1 = R_2 = R$ in (3.5) which becomes

$$\beta\langle Q^2 \rangle = \frac{1}{8\pi}\kappa S \tag{3.6}$$

where here $S = 4\pi R^2$ is the sphere area, in agreement with the general formula[1]

$$\frac{\langle Q^2 \rangle}{S} = -\frac{1}{4}\int \mathrm{d}\mathbf{r}\, r \langle \rho(0)\rho(\mathbf{r})\rangle \tag{3.7}$$

where $\langle \rho(0)\rho(\mathbf{r})\rangle$ is the infinite-system charge correlation function, obtained by using in (3.2) the infinite-system function $K(0,\mathbf{r}) = -\beta \exp(-\kappa r)/r$.

If, on the contrary, $R_2 - R_1$ is also macrosopic, i.e if we now assume $\kappa(R_2 - R_1) \gg 1$ as well as $\kappa R_1 \gg 1$ in (3.5), we do check the expected spherical capacitor charge fluctuation (2.1) with capacitance (2.3), as well as the grand-canonical fluctuation (2.4). The limit (2.5) of a plane capacitor can also be taken.

The plane capacitor can also be studied directly. There are two parallel planes $x = 0$ and $x = W$, thus separated by a distance W. The slab between the plates $0 < x < W$ is empty, while the Coulomb fluid fills the two semiinfinite regions $x < 0$ and $x > W$ outside the slab. In this geometry, it is possible to solve (3.2) for a function K which now is a function of x, x', and the component $\mathbf{y}$ of $\mathbf{r} - \mathbf{r}'$ along the plates (actually, one rather computes the Fourier transform of K with respect to $\mathbf{y}$). Then, one obtains the mean square charge per unit area on one plate of (infinite) area S as

$$\lim_{S\to\infty} \frac{\langle Q^2 \rangle}{S} = \int \mathrm{d}\mathbf{y} \int_{x<0} \mathrm{d}x \int_{x'<0} \mathrm{d}x' \langle \rho(x,\mathbf{y})\rho(x',0)\rangle \tag{3.8}$$

The result is

$$\lim_{S\to\infty}\frac{\beta\langle Q^2\rangle}{S}=\frac{\kappa}{4\pi(2+\kappa W)} \tag{3.9}$$

If $W=0$, one retrieves (3.6). If W is macroscopic, i.e. $\kappa W\gg 1$, one retrieves (2.5).

The above considerations also apply to the case of a one-component plasma (also called jellium), i.e. a system made of mobile point charges of one species, with number density n and charge q, in a uniform neutralizing background. The Debye wave number reduces to $\kappa=(4\pi\beta nq^2)^{1/2}$. However, for classical jellium, the grand canonical partition function is a convergent series only if the background is kept fixed while one sums over the particle number [5]. The same kind of prescription should be used here: regions A and B are assumed allowed to exchange particles, but the backgrounds keep a fixed charge density.

4 TWO-DIMENSIONAL COULOMB SYSTEMS

4.1 General Properties

In the two-dimensional systems discussed in this Section, the electric potential created at $\mathbf{r}$ by a unit charge at the origin is $-\ln(r/L)$ where L is some fixed length. This choice of a two-dimensional solution of the Poisson equation often makes these systems good toy models for mimicking three-dimensional systems with the usual $1/r$ potential. One of the advantages of working in two dimensions is the existence of exactly solvable models. For avoiding any confusion, it should be stressed that these toy models do *not* describe "real" charged particles such as electrons, which, even when confined in a surface, still interact by the $1/r$ law.

In the simple case of a circular capacitor, made of an inner circular conductor of radius R_1 and an outer circular conductor of radius R_2, macroscopic two-dimensional electrostatics says that the capacitance is

$$C=\frac{1}{\ln\frac{R_2}{R_1}} \tag{4.1}$$

C goes to 0 as $R_2\to\infty$.[2] More generally, the macroscopic capacitance of one finite conductor of any shape vanishes. This is because bringing from infinity an additional charge q onto a conductor of characteristic size R, already carrying a charge Q, would cost an energy of order $qQ\int_R^\infty \mathrm{d}r/r$ which is infinite.

It will however be shown below that, for the present problem of charge fluctuations, the mimicking of three-dimensional systems which might be expected at first glance does *not* always occur, because, in some cases, although all relevant lengths are macroscopic, nevertheless macroscopic electrostatics cannot be used. In the case of a circular capacitor, $\langle Q^2\rangle$ may not be given by using (4.1) in (2.1).

[2]In a previous paper[6], another definition of the capacitance of a disk was used. The present one (the limit of the capacitance of a circular capacitor when the outer conductor recedes to infinity) is more appropriate here.

4.2 Weak-Coupling Limit

The two-dimensional case is very similar to the three-dimensional one. We now consider the simple geometry of a circular capacitor replacing the concentric spheres by concentric circles: The regions $r < R_1$ and $r > R_2$ are occupied by the Coulomb fluid, while the region $R_1 < r < R_2$ is empty. The Debye-Hückel theory is again used, with 4π replaced by 2π in the definition of the Debye wave number κ and in the r.h.s. of (3.2). The detail of the calculation is given in Appendix A. The result is

$$\beta\langle Q^2\rangle = \frac{\kappa R_1}{\frac{I_0(\kappa R_1)}{I_1(\kappa R_1)} + \frac{R_1}{R_2}\frac{K_0(\kappa R_2)}{K_1(\kappa R_2)} + \kappa R_1 \ln\frac{R_2}{R_1}} \tag{4.2}$$

where I_l and K_l are modified Bessel functions, while $\langle Q\rangle = 0$. If both R_1 and R_2 are macroscopic, i.e. if $\kappa R_1, \kappa R_2 \gg 1$, (4.2) becomes

$$\beta\langle Q^2\rangle = \frac{\kappa R_1}{1 + \frac{R_1}{R_2} + \kappa R_1 \ln\frac{R_2}{R_1}} \tag{4.3}$$

The mean square charge in a large circular subdomain of an infinite system is retrieved by taking $R_1 = R_2 = R$ in (4.3) which becomes

$$\beta\langle Q^2\rangle = \frac{\kappa}{4\pi} S \tag{4.4}$$

where $S = 2\pi R$ is the subdomain perimeter, in agreement with the two-dimensional analog[1] of (3.7):

$$\frac{\langle Q^2\rangle}{S} = -\frac{1}{\pi}\int \mathrm{d}\mathbf{r}\, r\langle\rho(0)\rho(\mathbf{r})\rangle \tag{4.5}$$

where $\langle\rho(0)\rho(\mathbf{r})\rangle$ is the infinite-system charge correlation function, obtained by using in (3.2) the infinite-system function $K(0,\mathbf{r}) = -\beta K_0(\kappa r)$.

If, on the contrary, $R_2 - R_1$ is also macroscopic, i.e. if $\kappa(R_2 - R_1) \gg 1$ as well as $\kappa R_1 \gg 1$, (4.3) becomes

$$\beta\langle Q^2\rangle = \frac{1}{\ln\frac{R_2}{R_1}} \tag{4.6}$$

in agreement with (2.1) and (4.1). In the limit $R_2 \to \infty$, there is no charge fluctuation.

4.3 Two-Component Plasma at $\Gamma = 2$

The two-dimensional two-component plasma is made of two species of point particles of opposite charges $\pm q$, interacting through the pair potential $\pm q^2 \ln(r/L)$. The dimensionless coupling constant is $\Gamma = \beta q^2$. The system is stable against collapse of pairs of oppositely charged particles for $\Gamma < 2$. Many exact results are now available for that model in its whole stability range[7–11]. However, the correlation functions are fully known only at $\Gamma = 2$, for a variety of plane[12–17] (or even curved) geometries (although, for a given fugacity, the density starts to diverge at $\Gamma = 2$, the truncated many-body distribution functions remain finite).

By an extension of previous calculations, the correlation functions at $\Gamma = 2$, in the circular capacitor geometry described in Section 4.2 are obtained in Appendix C. The fugacity z (which is the same for both species) appears through a parameter $m = 2\pi Lz$ which has the dimension of an inverse length and defines a microscopic scale (the bulk correlation length is $(2m)^{-1}$). Using these correlation functions in (3.3) gives

$$\langle Q^2 \rangle = q^2 \sum_{l=-\infty}^{\infty} \frac{[(X^2+l^2)I_l^2(X) - X^2 I_l'^2(X)][lK_l^2(\alpha X) - \alpha X K_l(\alpha X)K_l'(\alpha X)]}{[lI_l(X)K_l(\alpha X)(\alpha^l - \alpha^{-l}) - XI_l(X)K_l'(\alpha X)\alpha^{l+1} + XI_l'(X)K_l(\alpha X)\alpha^{-l}]^2} \tag{4.7}$$

where $X = mR_1$ and $\alpha = R_2/R_1$. By charge symmetry, $\langle Q \rangle = 0$.

The mean square charge in a large circular subdomain of radius R of an infinite system is retrieved by taking $X = mR$ and $\alpha = 1$ in (4.7) and using for the modified Bessel functions I_l and K_l the leading terms of their uniform asymptotic expansions[18], which are appropriate in the present case of a large argument X and an index l which may also be large. It is found that the sum on l can be replaced by an integral. The result is

$$\langle Q^2 \rangle = q^2 \frac{m}{8} S \tag{4.8}$$

where $S = 2\pi R$ is the subdomain perimeter, in agreement with the general formula (4.5), where the infinite-system charge correlation function[13, 14] $\langle \rho(0)\rho(\mathbf{r}) \rangle$ is (up to its here irrelevant δ term) $-2q^2[m^2/(2\pi)]^2[K_0^2(mr) + K_1^2(mr)]$.

If, on the contrary, not only R_1 but also $R_2 - R_1$ are macroscopic, i.e. $X \gg 1$ with $\alpha > 1$, because of the $\alpha^l, \alpha^{-l}, \alpha^{l+1}$ terms in the denominator of (4.7) only small values of $|l|$ contribute to the sum, and the modified Bessel functions can be simply replaced by the leading terms of their ordinary asymptotic expansions[18], valid for a large argument X or αX and a given index l. In the limit $X = mR_1 \to \infty$ at fixed $\alpha = R_2/R_1$, with $\beta q^2 = 2$ being taken into account, (4.7) becomes

$$\beta \langle Q^2 \rangle = 2 \sum_{l=-\infty}^{\infty} \frac{1}{\left(\alpha^{l+\frac{1}{2}} + \alpha^{-(l+\frac{1}{2})}\right)^2} \tag{4.9}$$

where $\alpha = R_2/R_1$. It is clear that, when α has a finite value, the sum in (4.9) cannot be replaced by an integral [which would reproduce (4.6)].

(4.9) is, at first sight, a very surprising result. It does *not* reproduce the value (4.6) expected on the basis of macroscopic electrostatics. On second thought, one sees that the l.h.s. of (4.9) can be rewritten as $2\langle Q^2 \rangle / q^2$ since here $\Gamma = \beta q^2 = 2$, and therefore $\langle Q^2 \rangle / q^2$ is of order unity, which means that the fluctuation involves only a small number of particles. Thus, in spite of the fact that the relevant lengths R_1 and $R_2 - R_1$ are macroscopic, the number of involved particles is not, and (4.6) based on macroscopic electrostatics should not be expected to hold at $\Gamma = 2$, and more generally at any finite temperature. However, in the weak-coupling (i.e. high-temperature) limit considered in Section 4.2, $\beta q^2 \to 0$, $\langle Q^2 \rangle / q^2$ as given by (4.6) becomes large, and this result is consistent with macroscopic electrostatics, as it should.

It should be noted that, like (4.6), (4.9) indicates that there is no charge fluctuation in the limit $R_2 \to \infty$, i.e. for one macroscopic disk in the grand canonical ensemble. Only the charge $Q = 0$ contributes to the grand canonical distribution.

Another limit of interest is when $R_1 \to \infty$ for a fixed value of $W = R_2 - R_1$. One then obtains a two-dimensional "plane" capacitor. When this limit is approached, $\ln\alpha \sim W/R_1$ is small, and the sum in (4.9) can be replaced by the integral

$$\beta\langle Q^2\rangle = \frac{1}{2}\int_{-\infty}^{\infty} \frac{\mathrm{d}l}{\cosh^2(lW/R_1)} = \frac{2\pi R_1}{2\pi W} \tag{4.10}$$

This is the result expected on the basis of macroscopic electrostatics, the two-dimensional analog of (2.5) with now the plate area replaced by a plate length $2\pi R_1$ and the capacitance the two-dimensional one $C = 2\pi R_1/(2\pi W)$. Since $\langle Q^2\rangle/q^2$ now becomes large as $R_1 \to \infty$, macroscopic electrostatics should indeed hold. A direct derivation for a plane capacitor is feasible.

4.4 One-Component Plasma at $\Gamma = 2$

The two-dimensional one-component plasma is made of one species of point particles of charge q, interacting through the pair potential $-q^2\ln(r/L)$, in a neutralizing background of fixed charge density $-qn$. Far from the boundaries, the particle number density is n. The dimensionless coupling constant again is $\Gamma = \beta q^2$. Up to now, the system is exactly solvable[19] only at $\Gamma = 2$, in which case the correlation functions are known for a large variety of plane[20–23] (or even curved) geometries.

By an extention of previous calculations, the charge density and the correlation function at $\Gamma = 2$, in the circular capacitor geometry, are obtained in Appendix D. The background is fixed, while the two regions can freely exchange particles. There is no charge symmetry and the average charge $\langle Q\rangle$ on the inner disk does not automatically vanish. Using in (3.3) (modified for taking $\langle Q\rangle$ into account), the correlation function of Appendix D gives the charge fluctuation on the inner disk. We introduce the notations $Y_1 = \pi n R_1^2$ and $Y_2 = \pi n R_2^2$ and the incomplete gamma functions[24]

$$\gamma(l+1, Y_1) = \int_0^{Y_1} \mathrm{d}t\, \mathrm{e}^{-t} t^l \tag{4.11}$$

and

$$\Gamma(l+1+Y_2-Y_1, Y_2) = \int_{Y_2}^{\infty} \mathrm{d}t\, \mathrm{e}^{-t} t^{l+Y_2-Y_1} \tag{4.12}$$

The average charge on the inner disk is found to be

$$\langle Q\rangle = q\sum_{l=0}^{\infty} \frac{\gamma(l+1, Y_1)}{\gamma(l+1, Y_1) + D(l+1, Y_1, Y_2)} - qY_1 \tag{4.13}$$

where

$$D(l+1, Y_1, Y_2) = \exp(Y_1\ln Y_1 - Y_1 - Y_2\ln Y_2 + Y_2)\Gamma(l+1+Y_2-Y_1, Y_2) \tag{4.14}$$

The charge fluctuation on the inner disk is found to be

$$\langle Q^2\rangle - \langle Q\rangle^2 = q^2\sum_{l=0}^{\infty} \frac{\gamma(l+1, Y_1)D(l+1, Y_1, Y_2)}{[\gamma(l+1, Y_1) + D(l+1, Y_1, Y_2)]^2} \tag{4.15}$$

The integrands in (4.11) and (4.12) have a maximum (a sharp one when l is large) at $t = l$ and $t = l + Y_2 - Y_1$, respectively. Therefore, $\gamma(l+1, Y_1)$ becomes small when $l > Y_1$, while $\Gamma(l+1+Y_2-Y_1, Y_2)$ and thus $D(l+1, Y_1, Y_2)$ become small when $l < Y_1$. The summand in (4.15) has a maximum near $l = Y_1$.

The results for a large circular subdomain of radius R of an infinite system are retrieved by taking $R_1 = R_2 = R$, i.e $Y_1 = Y_2 = \pi n R^2$. Then (4.13) gives $\langle Q \rangle = 0$. Using for the incomplete gamma functions (4.11) and (4.12) the Tricomi asymptotic representation[24]

$$\gamma(l+1, Y) \sim \Gamma(l+1)[\frac{1}{2} + \pi^{-1/2}\mathrm{Erf}(\frac{Y-l}{(2Y)^{1/2}})] \tag{4.16}$$

where Γ is the complete gamma function and Erf is the error function (this representation is appropriate when Y is large and l is close to Y), it is found that the sum on l in (4.15) can be replaced by an integral. The result is

$$\langle Q^2 \rangle = q^2 \frac{n^{1/2}}{2\pi} S \tag{4.17}$$

where $S = 2\pi R$ is the subdomain perimeter, again in agreement with the general formula (4.5), where the infinite-system charge correlation function[20] $\langle \rho(0)\rho(\mathbf{r}) \rangle$ now is (up to its here irrelevant δ term) $-q^2 n^2 \exp(-\pi n r^2)$.

We now turn to the case when not only R_1 but also $R_2 - R_1$ are macroscopic, i.e. $Y_1 \gg 1$ with $\alpha = R_2/R_1 = (Y_2/Y_1)^{1/2} > 1$. The average charge (4.13) does not seem to have a simple expression. For investigating the charge fluctuation (4.15), it is convenient to rewrite the summand in it as $[(D/\gamma)^{1/2} + (\gamma/D)^{1/2}]^{-2}$. As it will be seen a posteriori, now the relevant values of $l - Y_1$ in the sum (4.15) are only of the order of a few units (rather than of the order of $Y_1^{1/2}$) and therefore the Erf term in the Tricomi representation (4.16) can be omitted and the incomplete gamma functions just replaced by half the corresponding complete one. Furthermore, one can use for these complete gamma functions the Stirling asymptotic expansions $\Gamma(1+l) \sim \exp[l \ln l - l + (1/2)\ln(2\pi l)]$ and the similar one for $\Gamma(l+1+Y_2-Y_1)$. Neglecting terms of order $1/Y_1$ or smaller, for a given value of $l - Y_1$, in $\ln(D/\gamma)$, one finds

$$\frac{D(l+1, Y_1, Y_2)}{\gamma(l+1, Y_1)} = \exp[(l - Y_1 + \frac{1}{2})\ln(Y_2/Y_1)] \tag{4.18}$$

Finally, one can shift the summation index in (4.15), replacing $l - Y_1$ by $l - \bar{Y}_1$, where $\bar{Y}_1$ is the non-integer part of Y_1 such that $0 \le \bar{Y}_1 < 1$, and extend the summation to $-\infty$ in the present large-Y_1 limit. Thus, $\beta q^2 = 2$ being taken into account,

$$\beta(\langle Q^2 \rangle - \langle Q \rangle^2) = 2 \sum_{l=-\infty}^{\infty} \frac{1}{\left(\alpha^{l-\bar{Y}_1+\frac{1}{2}} + \alpha^{-(l-\bar{Y}_1+\frac{1}{2})}\right)^2} \tag{4.19}$$

where $\alpha = R_2/R_1$. The form of (4.19) justifies a posteriori our above statement that the relevant values of $l - Y_1$ in (4.15) are only of the order of a few units.

Here too, for a finite value of α, the sum (4.19) cannot be replaced by an integral and does not reproduce the value (4.6) expected on the basis of macroscopic electrostatics.

The reason is the same as in the case of the two-component plasma: the fluctuation involves only a small number of particles. If $\bar{Y}_1 = 0$, i.e if the background charge $-qn\pi R^2$ on the inner disk is an integer number of elementary charges $-q$, the fluctuation (4.19) for the one-component plasma is the same as the fluctuation (4.9) for the two-component plasma; we have no explanation to offer.

The case of one disk alone, in the grand canonical ensemble (with a fixed background), is obtained by taking the limit $R_2 \to \infty$. Now the average charge on the disk has a simple form. For obtaining it, it is convenient to rewrite the summand in (4.13) as $[1+(D/\gamma)]^{-1}$, where now, from (4.18), $D/\gamma = 0$ if $l < Y_1 - (1/2)$, $D/\gamma = 1$ if $l = Y_1 - (1/2)$, and $D/\gamma = \infty$ if $l > Y_1 - (1/2)$. Therefore, since the summation index l is an integer,

$$\begin{aligned} \langle Q \rangle &= -q\bar{Y}_1 \quad \text{if } \bar{Y}_1 < \frac{1}{2} \\ \langle Q \rangle &= 0 \quad \text{if } \bar{Y}_1 = \frac{1}{2} \\ \langle Q \rangle &= q(1-\bar{Y}_1) \quad \text{if } \bar{Y}_1 > \frac{1}{2} \end{aligned} \tag{4.20}$$

For the behavior of the fluctuation (4.19), in the limit $\alpha \to \infty$, two cases have to be distinguished.

$$\begin{aligned} \langle Q^2 \rangle - \langle Q \rangle^2 &= 0 \quad \text{if } \bar{Y}_1 \neq \frac{1}{2} \\ \langle Q^2 \rangle - \langle Q \rangle^2 &= \frac{q^2}{4} \quad \text{if } \bar{Y}_1 = \frac{1}{2} \end{aligned} \tag{4.21}$$

These results have a simple interpretation: $-qY_1 = -q\pi n R_1^2$ is the negative background charge. Taking into account a (necessarily integer) number of positive particles gives a total charge (background plus particles) Q. If $\bar{Y}_1 \neq 1/2$, only one value of Q contributes to the grand canonical ensemble: the one which corresponds to the smallest possible value of $|Q|$. If however $\bar{Y}_1 = 1/2$, this smallest possible value is $|Q| = (q/2)$ which corresponds to two possibilities $Q = \pm(q/2)$ with equal probabilities.

The two-dimensional "plane capacitor" limit ($R_1 \to \infty$ for a fixed value of $W = R_2 - R_1$) is the same as in the case of a two-component plasma. As this limit is approached, the sum in (4.19) can be replaced by an integral, and $\langle Q \rangle = 0$ because of the geometrical symmetry between the two flat plates. Again, the result is (4.10), in agreement with macroscopic electrostatics.

5 SUMMARY AND CONCLUSION

For studying the charge fluctuations on a macroscopic but finite classical Coulomb system in the grand canonical ensemble, i.e. when the system is allowed to exchange particles with a reservoir, in a first step we have considered a capacitor: one electrode is the finite system under consideration, the other electrode surrounds the first one at some distance of it and extends to infinity. Both electrodes are assumed to be made of the same Coulomb fluid. The capacitor is short-circuited, which means that the two electrodes can

freely exchange particles. When the external electrode recedes to infinity, the internal one becomes one Coulomb system in a grand canonical ensemble. For actual calculations, the simple geometry of a spherical capacitor has been chosen.

For a three-dimensional system, there is no surprise. A short-circuited capacitor of capacitance C can be considered as an electric oscillator, and it is rather natural to state that its average energy $\langle Q^2 \rangle/2C$ is $(1/2)T$: this gives (2.1). This general formula is supported by the derivation of Section 2, on the basis of linear response theory and macroscopic electrostatics. It has been checked in Section 3, in the Debye-Hückel theory.

Two-dimensional Coulomb systems (with a logarithmic interaction) are more tricky. Since some of them are exactly solvable, it was tempting to test on them the general formula (2.1) for the charge fluctuations. It has been a surprise for the author that this general formula is *not* valid for a circular capacitor at some finite temperature. On second thought, one realizes that the charge fluctuations involve only a small number of particles, and therefore one should not expect the validity of a macroscopic formula.

In the limiting case of one disk alone in a grand canonical ensemble (i.e. the disk is allowed to exchange particles with a reservoir at infinity), in general there is no charge fluctuation and the charge Q is such that $|Q|$ has the smallest possible value (0 when possible, a fraction of the elementary charge q in the case of a one-component plasma with a background charge which is not an integer number of elementary charges $-q$). An exception is when the background charge of the one-component plasma is of the form$-[(N+(1/2)]q$ with N integer. Then both values $Q = \pm(1/2)q$ are equally probable. We cannot explain why there is no charge fluctuation in the cases when the smallest possible value of the total charge is 0. Indeed, in these cases, bringing another elementary charge $\pm q$ from infinity would cost only a finite energy and one would expect $Q = \pm q$ to contribute to the grand canonical ensemble. That these values $Q = \pm q$ do not contribute might be a special feature of the solvable models at $\Gamma = 2$. We just do not know.

Another tricky feature of two-dimensional Coulomb systems is that the two-dimensional Coulomb potential $-\ln(r/L)$ does not vanish at infinity, if the length L is finite. As discussed in Appendix B, for obtaining sensible results in the Debye-Hückel theory, it is necessary to take the limit $L \to \infty$.

APPENDIX A: DEBYE-HÜCKEL THEORY IN A SPHERICAL OR CIRCULAR CAPACITOR

In the spherical capacitor geometry, the solution of (3.2) can be expanded in Legendre polynomials $P_l(\cos\theta)$ (where θ is the angle between $\mathbf{r}$ and $\mathbf{r}'$) in the form

$$K(\mathbf{r}, \mathbf{r}') = \sum_{l=0}^{\infty} k_l(r, r') P_l(\cos\theta) \tag{A.1}$$

When this expansion is used in (3.1) and (3.3), only the term $l = 0$ survives the angular integrations. Thus, we only need the function $k_0(r, r')$. The solution of (3.2) for an infinite system is $K(\mathbf{r}, \mathbf{r}') = -\beta \exp(|\mathbf{r} - \mathbf{r}'|)/|\mathbf{r} - \mathbf{r}'|$ and its $l = 0$ part is $-\beta \sinh(\kappa r_<) \exp(-\kappa r_>)/(\kappa r r')$, where $r_<$ ($r_>$) is the smaller (the larger) of r and r'.

In the present geometry, there are additional "reflected" and "transmitted" terms. When the source point $\mathbf{r}'$ is in the inner sphere ($r' < R_1$), the solution is of the form

$$\begin{aligned} k_0(r,r') &= -\frac{\beta}{\kappa r r'}[\sinh(\kappa r_<)\exp(-\kappa r_>) + a\sinh(\kappa r)\sinh(\kappa r')] \quad (r,r' < R_1) \\ k_0(r,r') &= -\beta\frac{\sinh(\kappa r')}{r'}[\frac{b}{\kappa r} + c] \quad (r' < R_1,\ R_1 < r < R_2) \\ k_0(r,r') &= -\frac{\beta d}{\kappa r r'}\sinh(\kappa r')\exp(-\kappa r) \quad (r' < R_1,\ r > R_2) \end{aligned} \tag{A.2}$$

This solution has the appropriate singularity at $r = r'$, is otherwise regular at $r = 0$ and $r' = 0$, and goes to 0 when $r \to \infty$. The four coefficients a, b, c, d are to be determined by the requirements that $k_0(r,r')$ and $\partial k_0(r,r')/\partial r$ be continuous at $r = R_1$ and $r = R_2$. In particular, one finds

$$a = \frac{\exp(-\kappa R_1)(\kappa R_2 - \kappa R_1)}{(1 + \kappa R_2 - \kappa R_1)\cosh(\kappa R_1) + \sinh(\kappa R_1)} \tag{A.3}$$

Using the first equation (A.2) and (A.3) in (3.1) and (3.3) gives (3.4).

In two dimensions, in the circular capacitor geometry, the calculation is very similar to the above one. The expansion (A.1) is replaced by

$$K(\mathbf{r},\mathbf{r}') = \sum_{l=0}^{\infty} k_l(r,r')\cos(l\theta) \tag{A.4}$$

where the $l = 0$ part is, in terms of modified Bessel functions I_0 and K_0, when $r' < R_1$,

$$\begin{aligned} k_0(r,r') &= -\beta[I_0(\kappa r_<)K_0(\kappa r_>) + aI_0(\kappa r)I_0(\kappa r')] \quad (r,r' < R_1) \\ k_0(r,r') &= -\beta I_0(\kappa r')[b\ln(\kappa r) + c] \quad (r' < R_1,\ R_1 < r < R_2) \\ k_0(r,r') &= -\beta d I_0(\kappa r')K_0(\kappa r) \quad (r' < R_1,\ r > R_2) \end{aligned} \tag{A.5}$$

By the same method as above, the coefficient a is found as

$$a = \frac{-K_0(\kappa R_1) + \frac{R_1}{R_2}\frac{K_0(\kappa R_2)}{K_1(\kappa R_2)}K_1(\kappa R_1) - \kappa R_1 K_1(\kappa R_1)\ln\frac{R_1}{R_2}}{I_0(\kappa R_1) + \frac{R_1}{R_2}\frac{K_0(\kappa R_2)}{K_1(\kappa R_2)}I_1(\kappa R_1) - \kappa R_1 I_1(\kappa R_1)\ln\frac{R_1}{R_2}} \tag{A.6}$$

Using the first equation (A.5) and (A.6) in (3.1) and (3.3), and the Wronskian of the Bessel functions, gives (4.2).

APPENDIX B: DEBYE-HÜCKEL THEORY IN A FINITE SYSTEM

In Section 3 and Appendix A, the Debye-Hückel differential equation (3.2) was written and solved in the spherical capacitor geometry; with minor modifications, the same approach holds in the two-dimensional case, for a circular capacitor, as discussed in Section 4.2 and Appendix A. In these geometries, the Coulomb fluid extends to infinity in

the region $r > R_2$ and it is obvious that the boundary condition should be $K(\mathbf{r}, \mathbf{r}') \to 0$ as $r \to \infty$) (In this approach perfect screening is globally satisfied: The charge of one particle plus the charge it induces in the two conductors sum to zero. However, there is no perfect screening if one takes into account only the charge of a particle sitting on one of the conductors and the charge it induces on that conductor only, and this gives rise to a charge fluctuation on each conductor). The case of one sphere or disk in the grand canonical ensemble was obtained by taking the limit $R_2 \to \infty$ at the end of the calculation.

The question arises of how to formulate the Debye-Hückel theory in the grand canonical ensemble, directly starting with only a sphere or a disk of radius R. What is the boundary condition to be imposed at $r = R$? Choquard et al.[25] have already investigated this problem. Nevertheless, we revisit it, hoping to clarify some delicate points.

We start with the three-dimensional case. An unambiguous way of formulating the Debye-Hückel theory is to start with a full diagrammatic expansion[26] in the grand canonical ensemble, and to make a topological reduction replacing the fugacity (fugacities) by the density (densities). The Debye-Hückel correlation function is obtained by resumming a class of diagrams (the chain diagrams), or, equivalently, by taking for the function K in eq.(3.1) the solution of the integral equation

$$K(\mathbf{r}, \mathbf{r}') = -\frac{\beta}{|\mathbf{r} - \mathbf{r}'|} - \frac{\kappa^2}{4\pi} \int \mathrm{d}\mathbf{r}'' \frac{1}{|\mathbf{r} - \mathbf{r}''|} K(\mathbf{r}'', \mathbf{r}') \tag{B.1}$$

This integral equation can also be seen as the Ornstein-Zernicke equation in which the direct correlation function between particles of species a and b is approximated by $-\beta$ times their bare Coulomb interaction $q_a q_b / |\mathbf{r} - \mathbf{r}'|$. In a finite system, the densities n_a in $\kappa^2 = 4\pi\beta \sum_a n_a q_a^2$ are position-dependent near the boundaries. However, for the large-size systems considered here, this effect can be neglected and κ^2 will be taken as a constant in the whole system.

By taking the Laplacian of both sides of the integral equation (B.1), one obtains the partial differential equation (3.2). However, the integral equation provides the boundary condition to be used in (3.2). In the presently studied case of a sphere of radius R, we can use the expansion (A.1) in Legendre polynomials. For brevity we only consider the $l = 0$ part. The integral equation (B.1) gives for the boundary condition on the surface $r = R$ (with $r' < R$)

$$k_0(R, r') = -\frac{\beta}{R}[1 + \frac{\kappa^2}{\beta} \int_0^R \mathrm{d}r''\, r''^2 k_0(r'', r')] \tag{B.2}$$

Using in (B.2) the general form (A.2) of the solution of the partial differential equation (3.2) (with now $r, r' \leq R$) and performing the integral determines the coefficient a as

$$a = \frac{\exp(-\kappa R)}{\cosh(\kappa R)} \tag{B.3}$$

(The square bracket in the r.h.s. of (B.2) does not vanish: there is no perfect screening on the sole sphere). This directly obtained result (B.3) is identical to the limit of the spherical capacitor a coefficient (A.3) (with $R_1 = R$) as $R_2 \to \infty$. One recovers the mean

square charge (2.4). The existence of a fluctuation confirms that the formulation of the Debye-Hückel theory by the integral equation (B.1) is a grand canonical one.

The two-dimensional case is more tricky. The bare Coulomb interaction between two unit charges now is $-\ln(|\mathbf{r}-\mathbf{r}'|/L)$, with L some fixed length. The integral equation now is

$$K(\mathbf{r},\mathbf{r}') = \beta \ln \frac{|\mathbf{r}-\mathbf{r}'|}{L} + \frac{\kappa^2}{2\pi}\int \mathrm{d}\mathbf{r}'' \ln \frac{|\mathbf{r}-\mathbf{r}''|}{L} K(\mathbf{r}'',\mathbf{r}') \tag{B.4}$$

with $\kappa^2 = 2\pi\beta \sum_a n_a q_a^2$. The $l=0$ part of the expansion (A.4) obeys the boundary condition (with $r' < R$)

$$k_0(R,r') = \beta \ln \frac{R}{L}[1 + \frac{\kappa^2}{\beta}\int_0^R \mathrm{d}r''\, r'' k_0(r'',r')] \tag{B.5}$$

Using in (B.5) the general form (A.5) of the solution of the two-dimensional analog of the partial differential equation (3.2) (with now $r, r' \le R$) and performing the integral determines the coefficient a as

$$a = \frac{\kappa R K_1(\kappa R)\ln\frac{R}{L} + K_0(\kappa R)}{\kappa R I_1(\kappa R)\ln\frac{R}{L} - I_0(\kappa R)} \tag{B.6}$$

This result (B.6) is unacceptable as it stands, since it depends on the arbitrary length L, which only determines the zero of the potential and should not enter the correlation functions. Actually, using a Coulomb interaction $-\ln(r/L)$ which does not vanish at infinity causes difficulties at many places. For instance, the Coulomb energy of a macroscopic disk of radius R, carrying the macroscopic charge Q near its circumference, would be $(1/2)Q^2\ln(L/R)$, negative if $R > L$. Thus, in the grand canonical ensemble, configurations of infinite $|Q|$ would be favored, causing the grand canonical partition function to diverge. This seems to indicate that the limit $L \to \infty$ (such that the zero of the potential recedes to infinity) should be taken in (B.6), which then becomes

$$a = \frac{K_1(\kappa R)}{I_1(\kappa R)} \tag{B.7}$$

This result (B.7) is identical to the limit of the circular capacitor a coefficient (A.6) (with $R_1 = R$) as $R_2 \to \infty$. It might even be noted that, if in (A.6) we take $R_1 = R$ and $R_2 = L \gg R$ and neglect the term of order $R_1/R_2 = R/L$ in both the numerator and the denominator, (B.6) is recovered. This is a further indication that the limit $L \to \infty$ should be taken in the case of a system made of one disk only. Now, there is perfect screening (the square bracket in the r.h.s. of (B.5) vanishes) and there is no charge fluctuation on the disk.

Still another way of dealing with a finite system, in three or two dimensions, would be to first assume that the whole space external to the system is filled with a medium of Debye wave number κ' and solve the partial differential equation (3.2) (or its two-dimensional analog) in the whole space, taking $\kappa^2(\mathbf{r}) = \kappa^2$ in the system and $\kappa^2(\mathbf{r}) = \kappa'^2$ outside, with the proper continuity conditions on the system boundary and $K(\mathbf{r},\mathbf{r}') \to 0$ as $r \to \infty$. One recovers the same results as above.

APPENDIX C: TWO-DIMENSIONAL TWO-COMPONENT PLASMA AT $\Gamma = 2$

For this exactly solvable model[14], the charge correlation function can be expressed in terms of Green functions $G_{++}(\mathbf{r},\mathbf{r}')$ and $G_{-+}(\mathbf{r},\mathbf{r}')$ as

$$\langle \rho(\mathbf{r})\rho(\mathbf{r}')\rangle = -2m^2q^2[|G_{++}(\mathbf{r},\mathbf{r}')|^2 + |G_{-+}(\mathbf{r},\mathbf{r}')|^2] + n(\mathbf{r})q^2\delta(\mathbf{r}-\mathbf{r}') \qquad \text{(C.1)}$$

where m is a rescaled fugacity such that $(2m)^{-1}$ is the bulk correlation length, and $n(\mathbf{r})$ is the total number density (actually, n is a divergent quantity, but it will be seen that this divergence causes no difficulty here). In (C.1), the charge symmetry of the model has been taken into account. We consider the circular capacitor geometry and the source point $\mathbf{r}'$ is in the inner disk ($r' < R_1$). For (C.1) to be the charge correlation function, $\mathbf{r}$ must be in a filled region ($r < R_1$ or $r > R_2$).

When $\mathbf{r}$ is in a filled region, the Green functions obey the equations[14]

$$(m^2 - \Delta)G_{++}(\mathbf{r},\mathbf{r}') = m\delta(\mathbf{r}-\mathbf{r}') \qquad \text{(C.2)}$$

and

$$G_{-+}(\mathbf{r},\mathbf{r}') = -\frac{\exp(\mathrm{i}\varphi)}{m}\left(\frac{\partial}{\partial r} + \frac{\mathrm{i}}{r}\frac{\partial}{\partial\varphi}\right)G_{++}(\mathbf{r},\mathbf{r}') \qquad \text{(C.3)}$$

where (r,φ) are the polar coordinates of $\mathbf{r}$. The solution of (C.2) is an expansion of the form

$$G_{++}(\mathbf{r},\mathbf{r}') = \frac{m}{2\pi}\sum_{l=-\infty}^{\infty}[I_l(mr_<)K_l(mr_>) + a_l I_l(mr')I_l(mr)]\exp[\mathrm{i}l(\varphi-\varphi')] \quad (r, r' < R_1)$$

$$G_{++}(\mathbf{r},\mathbf{r}') = \frac{m}{2\pi}\sum_{l=-\infty}^{\infty} d_l I_l(mr')K_l(mr)\exp[\mathrm{i}l(\varphi-\varphi')] \quad (r' < R_1,\ r > R_2) \qquad \text{(C.4)}$$

These expansions have the proper singularity at $\mathbf{r} = \mathbf{r}'$, are otherwise regular at $r = 0$ and $r' = 0$, and go to zero when $r \to \infty$. As to G_{-+}, (C.3) gives

$$G_{-+}(\mathbf{r},\mathbf{r}') = \frac{m}{2\pi}\sum_{l=-\infty}^{\infty}[I_l(mr')K_{l+1}(mr) - a_l I_l(mr')I_{l+1}(mr)]\exp[\mathrm{i}(l+1)\varphi - \mathrm{i}l\varphi'] \quad (r' < r < R_1)$$

$$G_{-+}(\mathbf{r},\mathbf{r}') = \frac{m}{2\pi}\sum_{l=-\infty}^{\infty} d_l I_l(mr')K_{l+1}(mr)\exp[\mathrm{i}(l+1)\varphi - \mathrm{i}l\varphi'] \quad (r' < R_1,\ r > R_2) \qquad \text{(C.5)}$$

When $\mathbf{r}$ is in the empty region $R_1 < r < R_2$ (where $m = 0$), as functions of $\mathbf{r}$, G_{++} depends only on $z = r\exp(\mathrm{i}\varphi)$ and G_{-+} depends only on $\bar{z} = r\exp(-\mathrm{i}\varphi)$. Thus, the expansions are of the forms

$$G_{++}(\mathbf{r},\mathbf{r}') = \frac{m}{2\pi}\sum_{l=-\infty}^{\infty} b_l I_l(mr')(mr)^l\exp[\mathrm{i}l(\varphi-\varphi')] \quad (r' < R_1,\ R_1 < r < R_2) \qquad \text{(C.6)}$$

and

$$G_{-+}(\mathbf{r},\mathbf{r}') = \frac{m}{2\pi} \sum_{l=-\infty}^{\infty} c_l I_l(mr')(mr)^{-(l+1)} \exp[\mathrm{i}(l+1)\varphi - \mathrm{i}l\varphi'] \quad (r' < R_1,\ R_1 < r < R_2) \tag{C.7}$$

The coefficients a_l, b_l, c_l, d_l are to be determined by the requirements that G_{++} and G_{-+} be continuous at $r = R_1$ and $r = R_2$. In particular, after having used some functional relations between Bessel functions, one finds

$$d_l = \Big[lI_l(X)K_l(\alpha X)(\alpha^l - \alpha^{-l}) - XI_l(X)K_l'(\alpha X)\alpha^{l+1} + XI_l'(X)K_l(\alpha X)\alpha^{-l}\Big]^{-1} \tag{C.8}$$

where $X = mR_1$ and $\alpha = R_2/R_1$.

For computing the mean square charge on the inner disk, using the perfect screening relation $\int \mathrm{d}\mathbf{r}\langle\rho(\mathbf{r})\rho(\mathbf{r}')\rangle = 0$, where the integral is on the whole space, it is convenient to rewrite (3.3) as

$$\langle Q^2\rangle = -\int_{r>R_2} \mathrm{d}\mathbf{r} \int_{r'<R_1} \mathrm{d}\mathbf{r}'\langle\rho(\mathbf{r})\rho(\mathbf{r}')\rangle \tag{C.9}$$

and to use (C.1), omitting the self part $nq^2\delta(\mathbf{r}-\mathbf{r}')$ which does not contribute to (C.9). The angular integrations are easily performed, by using the mutual orthogonality of the functions $\exp(\mathrm{i}l\varphi)$, and (C.9) becomes

$$\langle Q^2\rangle = 2m^4q^2 \sum_{l=-\infty}^{\infty} d_l^2 \int_0^{R_1} \mathrm{d}r'\, r' I_l^2(mr') \int_{R_2}^{\infty} \mathrm{d}r\, r[K_l^2(mr) + K_{l+1}^2(mr)] \tag{C.10}$$

where d_l is given by (C.8). After having performed the integrations in (C.10) and used some functional relations between Bessel functions, one obtains (4.7).

APPENDIX D: TWO-DIMENSIONAL ONE-COMPONENT PLASMA AT $\Gamma = 2$

For this exactly solvable model[20, 22], the charge density (including the background contribution) and the charge correlation function can be expressed in terms of a projector $P(\mathbf{r},\mathbf{r}')$

$$\langle\rho(\mathbf{r})\rangle = q[P(\mathbf{r},\mathbf{r}) - n] \tag{D.1}$$

and

$$\langle\rho(\mathbf{r})\rho(\mathbf{r}')\rangle - \langle\rho(\mathbf{r})\rangle\langle\rho(\mathbf{r}')\rangle = q^2[-|P(\mathbf{r},\mathbf{r}')|^2 + P(\mathbf{r},\mathbf{r})\delta(\mathbf{r}-\mathbf{r}')] \tag{D.2}$$

We consider the circular capacitor geometry. For (D.1) and (D.2) to be the charge density and correlation, respectively, $\mathbf{r}$ and $\mathbf{r}'$ must be in a filled region.

The electric potential $qV(r)$ created by the background will be needed. It obeys $\Delta V(r) = 2\pi n$ in the filled regions $r < R_1$ and $r > R_2$. It obeys $\Delta V(r) = 0$ in the empty region $R_1 < r < R_2$. At the boundaries $r = R_1$ and $r = R_2$, $V(r)$ and $\mathrm{d}V/\mathrm{d}r$ must be continuous. Up to an overall irrelevant additive constant,

$$V(r) = \frac{1}{2}\pi n r^2 \quad (r < R_1)$$

$$
\begin{aligned}
&V(r) = \frac{1}{2}\pi n R_1^2 + \pi n R_1^2 \ln \frac{r}{R_1} \quad (R_1 < r < R_2) \\
&V(r) = \frac{1}{2}\pi n (R_1^2 - R_2^2) + \pi n R_1^2 \ln \frac{R_2}{R_1} + \\
&\frac{1}{2}\pi n r^2 + \frac{1}{2}\pi n (R_1^2 - R_2^2) \ln \frac{r}{R_2} \quad (r > R_2)
\end{aligned} \tag{D.3}
$$

$P(\mathbf{r}, \mathbf{r}')$ is the projector on the functional space spanned by the functions $\psi_l(\mathbf{r}) = \exp[-V(r)][r \exp(\mathrm{i}\varphi)]^l$ ($l = 0, 1, 2, 3, \ldots$) (this definition of ψ_l holds in the filled regions $r < R_1$ and $r > R_2$, while $\psi_l(\mathbf{r}) = 0$ in the empty region $R_1 < r < R_2$):

$$
P(\mathbf{r}, \mathbf{r}') = \sum_{l=0}^{\infty} \frac{1}{C_l} \psi_l(\mathbf{r}) \bar{\psi}_l(\mathbf{r}') = \sum_{l=0}^{\infty} \frac{1}{C_l} \exp[-V(r) - V(r')] r^l r'^l \exp[\mathrm{i}l(\varphi - \varphi')] \tag{D.4}
$$

where C_l is the normalization constant

$$
\begin{aligned}
C_l &= \left(\int_{r<R_1} + \int_{r>R_2}\right) \mathrm{d}\mathbf{r} \, \exp[-2V(r)] r^{2l} \\
&= \frac{\pi}{(\pi n)^{l+1}} [\gamma(l+1, Y_1) + D(l+1, Y_1, Y_2)]
\end{aligned} \tag{D.5}
$$

where the functions γ and D are defined in (4.11) and (4.14).

The average charge on the inner disk is

$$
\langle Q \rangle = \int_{r<R_1} \mathrm{d}\mathbf{r} \langle \rho(\mathbf{r}) \rangle \tag{D.6}
$$

Using (D.1) and (D.4) in (D.6) and performing the integration gives (4.13).

The charge fluctuation on the inner disk is

$$
\langle Q^2 \rangle - \langle Q \rangle^2 = \int_{r<R_1} \mathrm{d}\mathbf{r} \int_{r'<R_1} \mathrm{d}\mathbf{r}' [\langle \rho(\mathbf{r}) \rho(\mathbf{r}') \rangle - \langle \rho(\mathbf{r}) \rangle \langle \rho(\mathbf{r}') \rangle] \tag{D.7}
$$

Using (D.2) and (D.4) in (D.7) and performing the integrations, using the mutual orthogonality of the functions $\exp(\mathrm{i}l\varphi)$, gives (4.15).

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